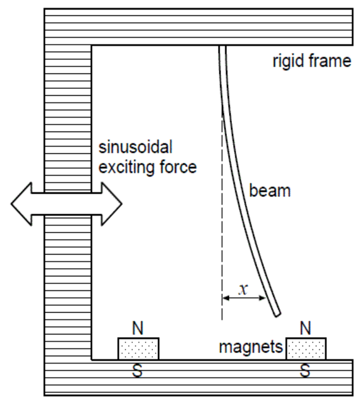

Numerical and Experimental Investigations into the nonlinear dynamics of a Magneto-elastic System

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Introduction

- Magneto-elastic systems : Motors, generators, mag-lev trains, magneto-elastic load cells.
- Simple system : Cantilevered beam between two magnets, with periodic forcing on the system
- Model : Forced, damped Duffing's oscillator (Holmes [1]): $\ddot{x} + \delta\dot{x} - \alpha x + \beta x^3 = \mathrm{P}\cos\omega t$
  - Displays chaotic phenomena, "strange attractor" structure on the Poincaré map.
  - Only models a double-well potential.
  - α and β have to be determined experimentally.
- Can we obtain a model based on physical parameters of the system? (e.g. Magnet spacing)

Objectives

- **Develop** a computational model for the governing ODE based on physical parameters.
- **Build** the magneto-elastic system in lab to compare computational results with experimental results (and also with theory of Duffing Oscillator)
- **Explore** the statics and dynamics of the system modeled by the computational model.
- **Search** for parameters that give rise chaotic motions as predicted by Duffing oscillator theory.

Background

- General model : Magnetic field induces body forces F_x ,F_y and moments C on the beam.
- PDE for beam displacement v(s,t) : (s = arc-length coordinate)

$$F_y - EIv'''' - C' + [Tv']' = m\left(\ddot{v} + \ddot{V}_0\right), \quad T = \int_s^L F_x\, ds.$$

- Nonlinearities : F_y , F_x and C depend on beam shape. $F_x = F_x(s,v)$, $F_y = F_y(s,v)$, $C = C(s,v)$

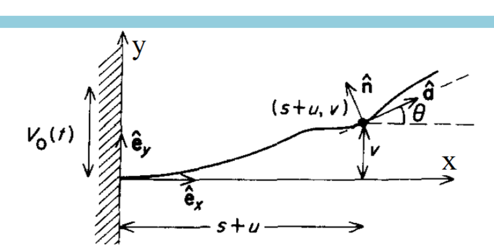
- Assume a single-mode approximation, $v(s,t) = \phi(s)a(t)$
- $\phi(s)$ is first spatial beam mode for linear scenario without magnets: $-EIv'''' = m\ddot{v}$

$$\phi(s) = c\left[K(sinh(ks) - sin(ks)) + (cosh(ks) - cos(ks))\right],$$

$$kL \approx 1.87510407, \quad K \approx -0.734096 \ , \ \mathrm{c} : \int_0^L \phi^2(s)\, ds = 1$$

- ODE for modal amplitude a(t)

$$\ddot{a} = F_{static}(a) - \ddot{V}_0 \int_0^L \phi\, ds.$$



$$F_{static}(a) = \frac{1}{m}\left[-\underbrace{\left(EI\int_0^L (\phi'')^2\, ds\right.}_{\text{Beam elasticity}} + \underbrace{\left.\int_0^L T(\phi')^2\, ds\right) a + \int_0^L F_y\phi\, ds + \int_0^L C\phi'\, ds}_{\text{Magnetic Field}}\right]$$

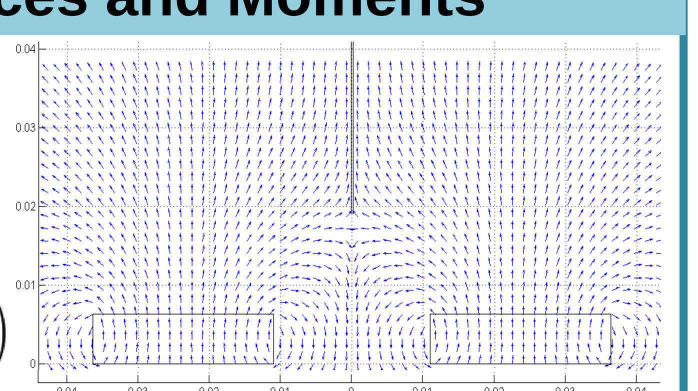
- Measure dynamics in terms of beam tip displacement $v_L(t) = a(t)\phi(L)$ instead of modal amplitude $a(t)$ since easier to do so experimentally.
- Assume linear viscous damping model with damping coefficient δ.
- For periodic forcing $V_0 = A_0 cos(\omega t)$ and $F_{static}(v_L) = \phi(L)F_{static}(a)$

$$\boxed{\ddot{v}_L = F_{static}(v_L) - \delta\dot{v}_L + P\,cos(\omega t)} \text{ with } P = \omega^2 A_0 \phi(L)\int_0^L \phi\, ds$$

- Cubic approximation : $F_{static}(v_L) \approx \alpha v_L - \beta v_L^3$ → Duffing oscillator
- Full model : Compute all terms in F_{static}

Computing Magnetic Forces and Moments

- Model cylindrical magnets as **ideal solenoids**.
- Use algorithm by Derby [2] to compute total magnetic field $\mathbf{B} = (B_x, B_y)$ at beam.
- For any point on the beam with angle θ against the vertical, the magnetization of the beam is



$$\mathbf{M} = \begin{pmatrix} M_x \\ M_y \end{pmatrix} = \frac{\chi A}{\mu_0 \mu_r}\begin{pmatrix} (1+\chi cos^2(\theta))B_x + \chi cos(\theta)sin(\theta)B_y \\ \chi cos(\theta)sin(\theta)B_x + (1+\chi sin^2(\theta))B_y \end{pmatrix}$$

$$\mathbf{F} = \mathbf{M}\cdot\nabla\mathbf{B} = \begin{pmatrix} M_x\frac{\partial B_x}{\partial x} + M_y\frac{\partial B_x}{\partial y} \\ M_x\frac{\partial B_y}{\partial x} + M_y\frac{\partial B_y}{\partial y} \end{pmatrix}, \quad \mathbf{C} = \mathbf{M}\times\mathbf{B}$$

Calculate derivatives of magnetic field using 4th-order central finite difference.

- Fix beam tip displacement → Fix beam shape in magnetic field. Partition nodes over beam.
- Compute all integrals related to $\phi(s)$ and its derivatives using trapezoidal rule.
- Compute forces and moments → Compute F_{static} for particular beam tip displacement.
- Repeat over an interval of beam tip displacements → Obtain $F_{static}(v_L)$.

Experimental Setup - Instrumentation

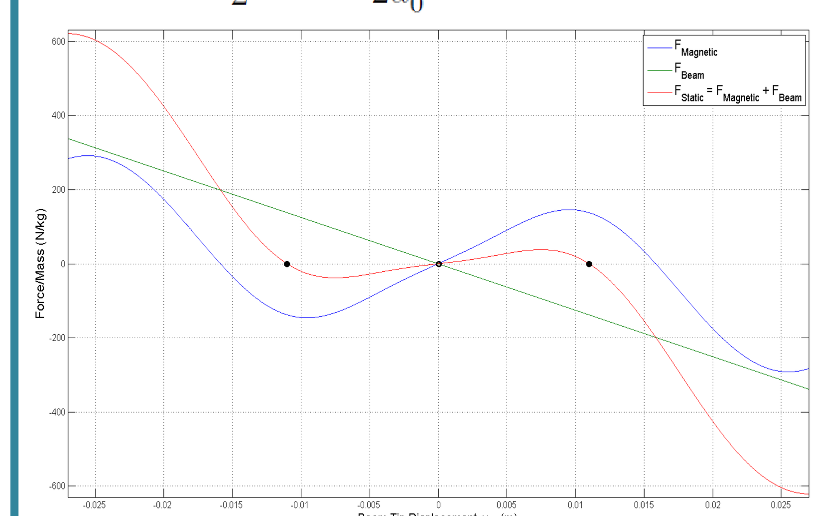
- Magnets : 0.21 Tesla magnetic field strength on surface.
- Beam : 1095 Steel. High yield strength (552 MPa), Young's Modulus E = 2.06*10^5 MPa.
- Beam motion visualized using Strain gauge (350 Ω) attached at root of beam.
- Wheatstone Quarter bridge (Temperature compensated) → Amplifier (Position signal) → Op-amp Differentiator (Velocity signal)
- 10-bit Analog-to-Digital Converter (ADC) on Arduino Uno → Laptop (Matlab) via USB Serial connection.
- Mean sampling rate = 620 ± 20 Hz. Uncertainty of ± 5 digital units in position, ± 10 digital units in velocity. (Digital range = 2^10 = 1024)

Free vibrations without magnets

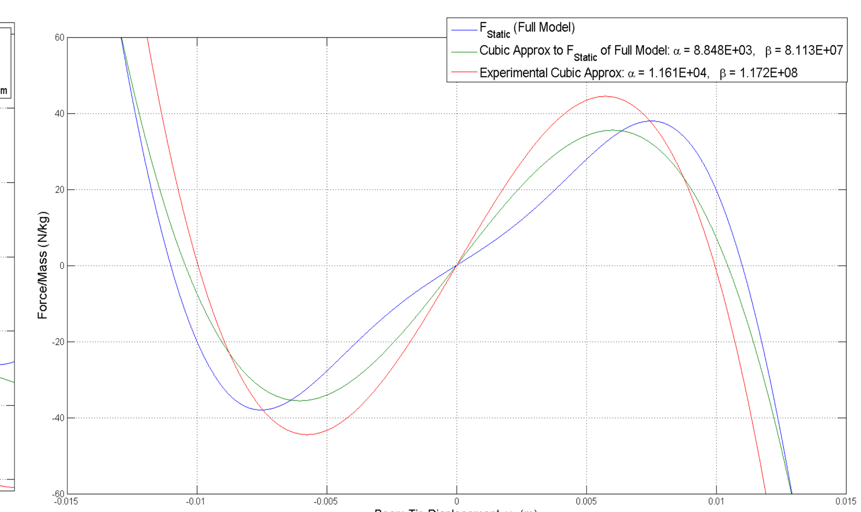
- Damping Ratio: 0.17308
- Natural Frequency:
  - 17.9 ± 0.3 Hz (Experiment) , 17.81 Hz (Theory)

Free vibrations with magnets

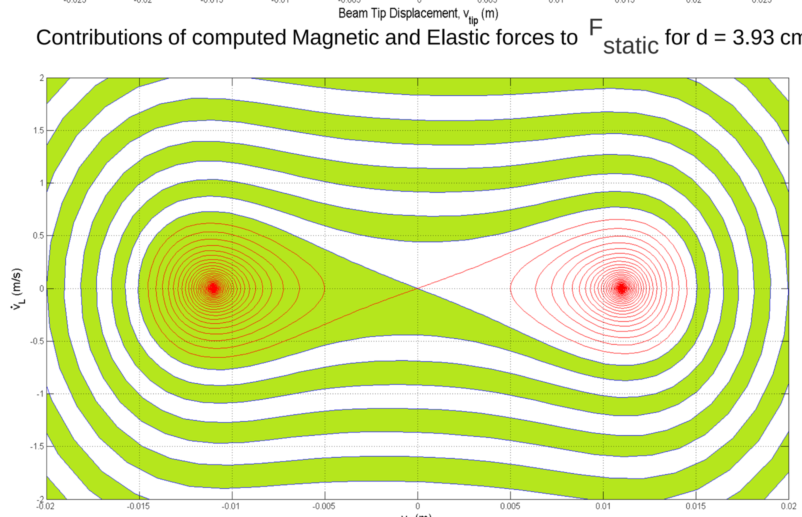
- Duffing oscillator : $\ddot{v_L} = \alpha v_L - \beta v_L^3 - \delta\dot{v_L}$
  - Equilibrium points at (0,0) (saddle) and $\left(\pm\sqrt{\frac{\alpha}{\beta}}, 0\right)$ (stable spiral sinks).
  - If we measure a buckled equilibrium position a_0, and a natural frequency ω_1 about that buckled position, then we can get a cubic approximation of F_{static} from experimental measurements as $\alpha = \frac{\omega_1^2}{2}, \ \beta = \frac{\omega_1^2}{2a_0^2}$. Experimentally, we measure a_0= 0.99 ± 0.02 cm and ω_1 = 24 ± 0.1 Hz.



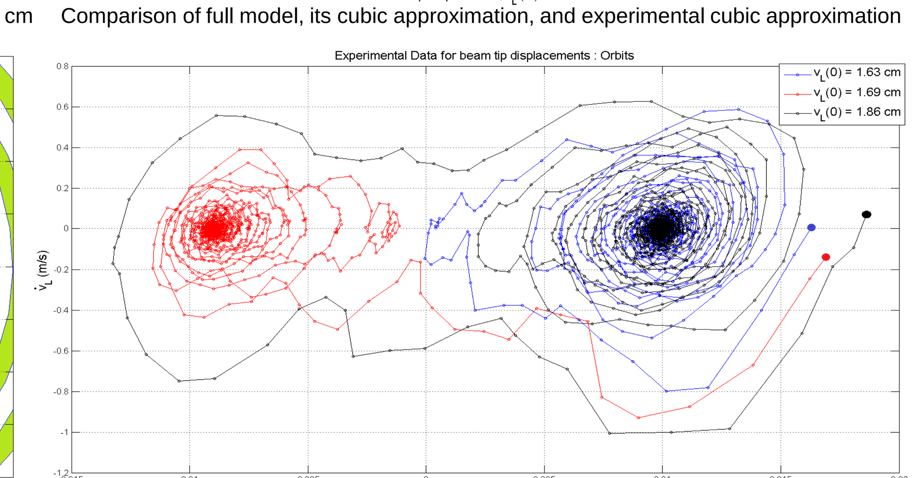
Contributions of computed Magnetic and Elastic forces to F_{static} for d = 3.93 cm



Comparison of full model, its cubic approximation, and experimental cubic approximation

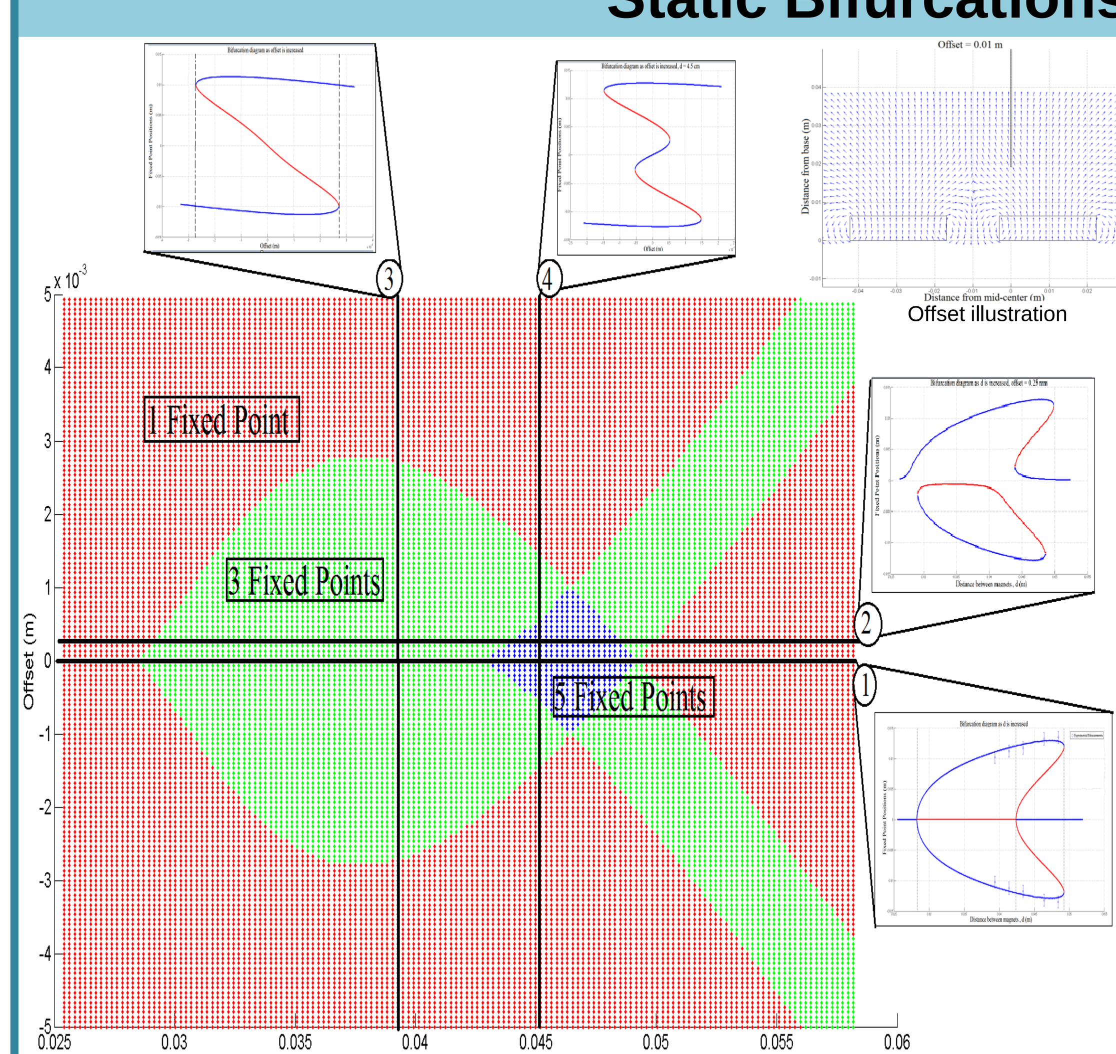


Basin of attraction of fixed points as computed using the computational model

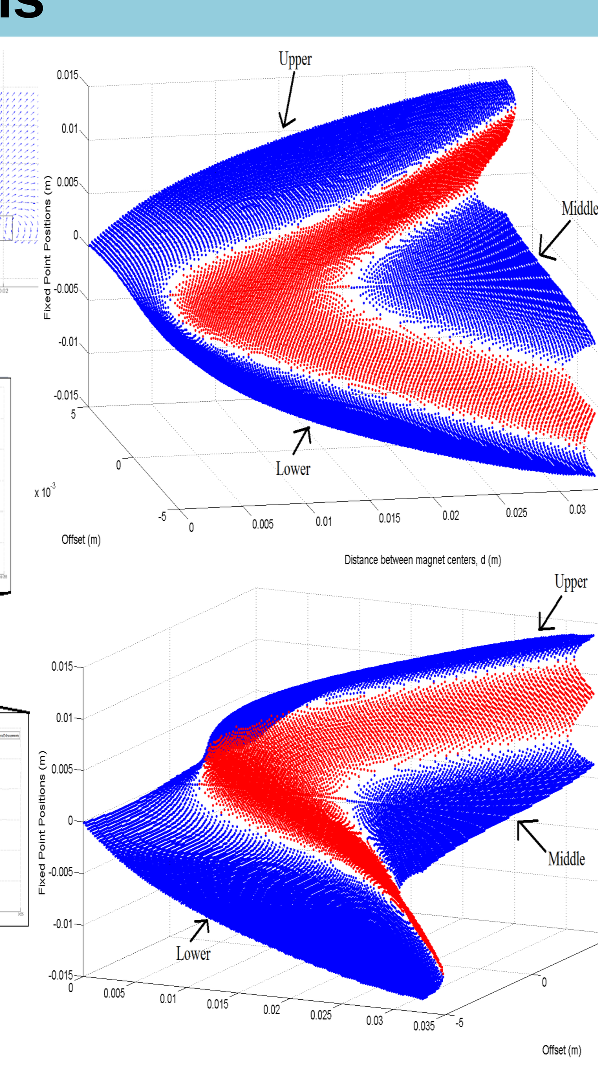


Orbits from experimental data for various initial displacements

Static Bifurcations



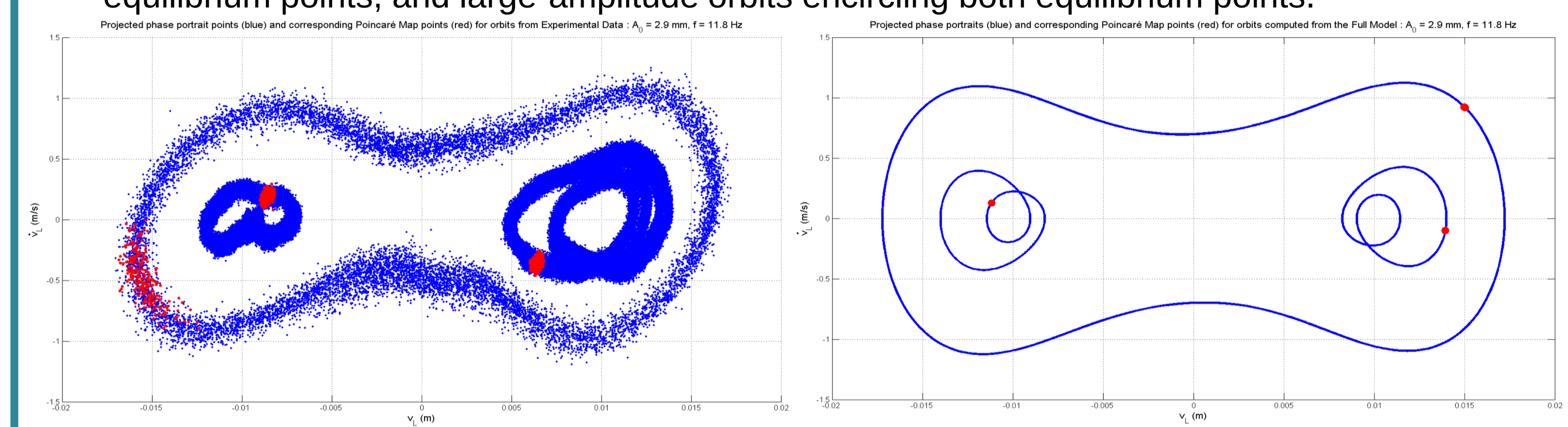
Bifurcation set over two parameters : Distance between magnets and beam offset



Surface of equilibria over two parameters

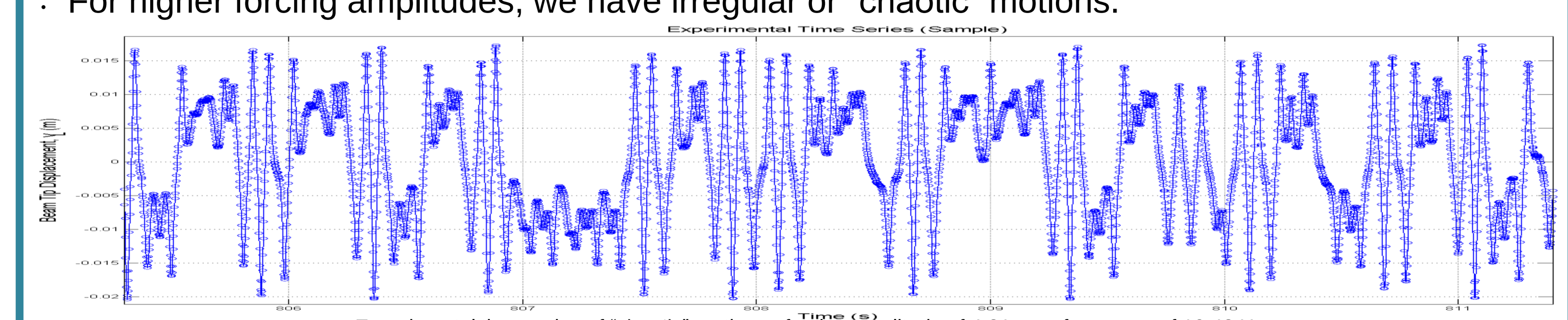
Forced Vibrations with Magnets

- For low forcing amplitudes, we find co-existence of small-amplitude orbits around one of the equilibrium points, and large-amplitude orbits encircling both equilibrium points.

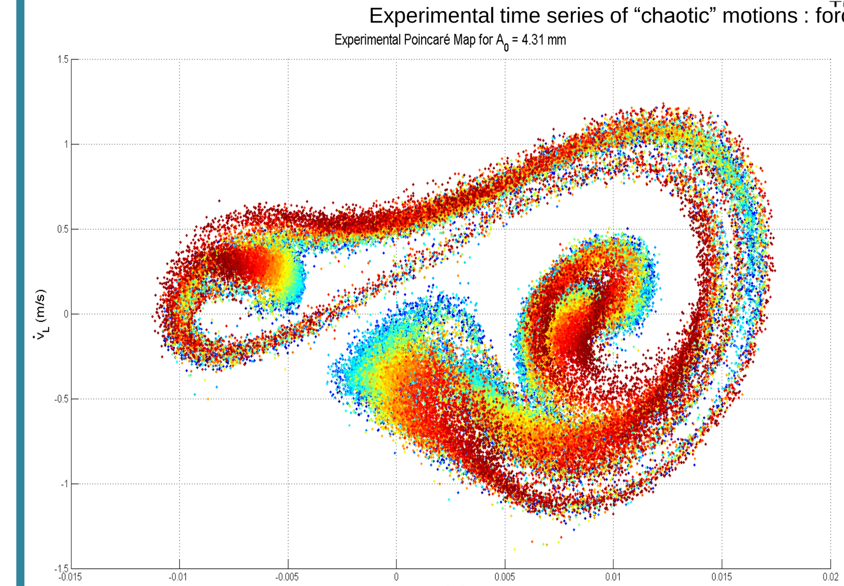


- Experimental and computational results for forcing amplitude of 2.9 mm, frequency of 11.8 Hz

- For higher forcing amplitudes, we have irregular or "chaotic" motions.

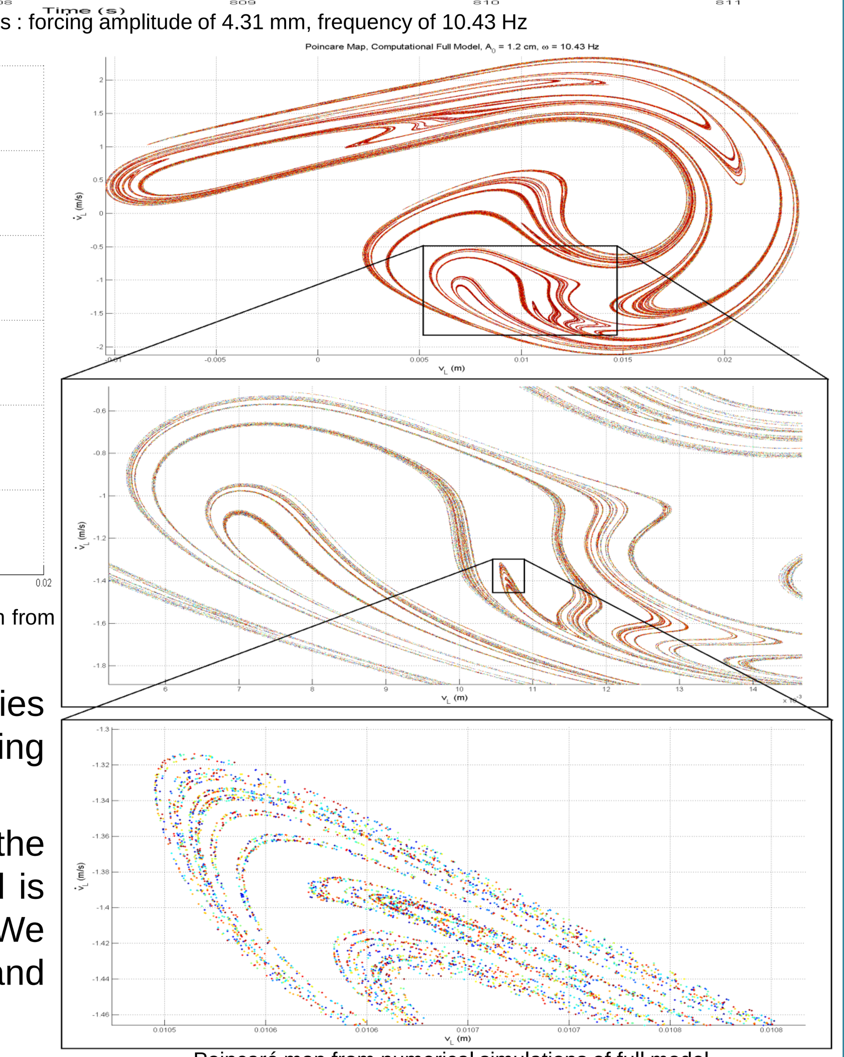


Experimental time series of "chaotic" motions : forcing amplitude of 4.31 mm, frequency of 10.43 Hz

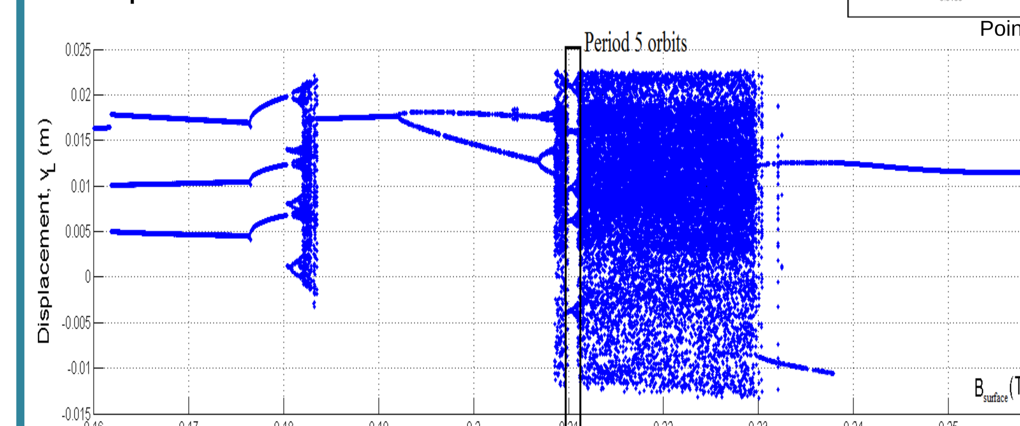


Poincaré map from experimental data. Color of points correspond to time taken from time series, with "colder" colors taken earlier in time

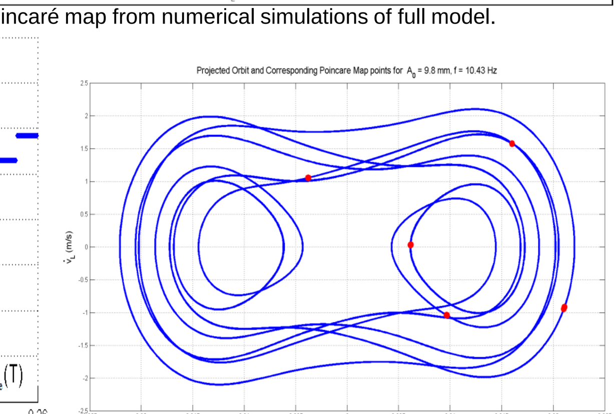
- Taking the Poincaré map from the time series (sample orbit on phase plane every forcing period) gives a "strange attractor" structure.
- This "strange attractor" is predicted by the theory for a Duffing oscillator as well [3], and is an extremely complicated fractal structure. We also observe this from experimental and computational results.



Poincaré map from numerical simulations of full model.



Bifurcation diagram of Poincaré map for increasing magnet field strengths.



Project orbit and Poincaré map points for a Period-5 orbit

Conclusions and Future Work

- The computational model matches experimental data and theoretical predictions qualitatively.
- The model offers more flexibility in predicting the behaviour of the system compared to the Duffing oscillator theory, as it is based on physical parameters.
- However, the numerical nature of the model makes it unamenable to analysis.
- An interesting variant is to consider what happens if the magnetic field strengths are varied periodically, and possibly in different phases. This can be done directly using the developed model with slight modifications, and experimentally this involves substituting the magnets with solenoids.

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Selected References

[1] F.C. Moon and P.J. Holmes 1979 *Journal of Sound and Vibration* **65** (2), 275-296. A Magnetoelastic Strange Attractor.
[2] N. Derby and S. Olbert 2010 *American Journal of Physics* **78** (3), 229. Cylindrical magnets and ideal solenoids.
[3] P.J. Holmes 1979 *Philosophical Transactions of the Royal Society* (London). A nonlinear oscillator with a strange attractor.